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The occurrence of negative added mass in free-surface problems involving submerged oscillating bodies

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Summary

The phenomenon of negative added mass is studied by considering the heave oscillations of a submerged vertical cylinder. Free-surface effects are shown to be important for the occurrence of negative added mass. Rapid changes in the added mass and damping, as functions of frequency of oscillation, often associated with this phenomenon are explained in terms of near-resonant standing waves above the body.

1. Introduction

In the linearised theory of water waves it is customary to (arbitrarily) decompose the hydrodynamic force on an oscillating body, due to its own motion, into components in phase with the acceleration and velocity of the body. These two components of the force are known respectively as the added-mass and damping coefficients for the body. Both are, in general, functions of the frequency of oscillation and, once known, allow prediction of the body motion in incident waves, provided the force on the *fixed* body in these waves is also known.

The damping coefficient is a measure of the energy flux in the waves radiating away from the oscillating body (Newman [7], p. 296) and is necessarily non-negative. For a deeply submerged body, the added-mass coefficient can be interpreted as the fluid mass accelerated by the body and is positive. However, when free-surface effects are important this need not be so. Ogilvie [10] calculated added-mass coefficients for a submerged horizontal cylinder. He found that when the depth of submergence is sufficiently small, compared to the diameter of the cylinder, the added mass is negative over a range of frequencies. Added-mass and damping curves calculated using the reformulation of Ogilvie's method by Evans, Jeffrey, Salter and Taylor [2] are presented in Figs. 1 and 2. Note the rapid changes in both quantities associated with the occurrence of negative added mass. Experimental confirmation of the behaviour was made by Chung [1] for cylinders with both square and circular cross-section. Negative added mass has also been found for floating bodies that enclose a region of the free surface; for example the floating torus described by Newman [8], and the two half-immersed cylinders studied by Wang and Wahab [14] as a model for the motion of a catamaran hull. Similar behaviour was obtained when two submerged cylinders were considered. An approximate solution for two vertical spaced rolling plates in two dimensions given by Srokosz and Evans [11] also produced negative added-mass coefficients.

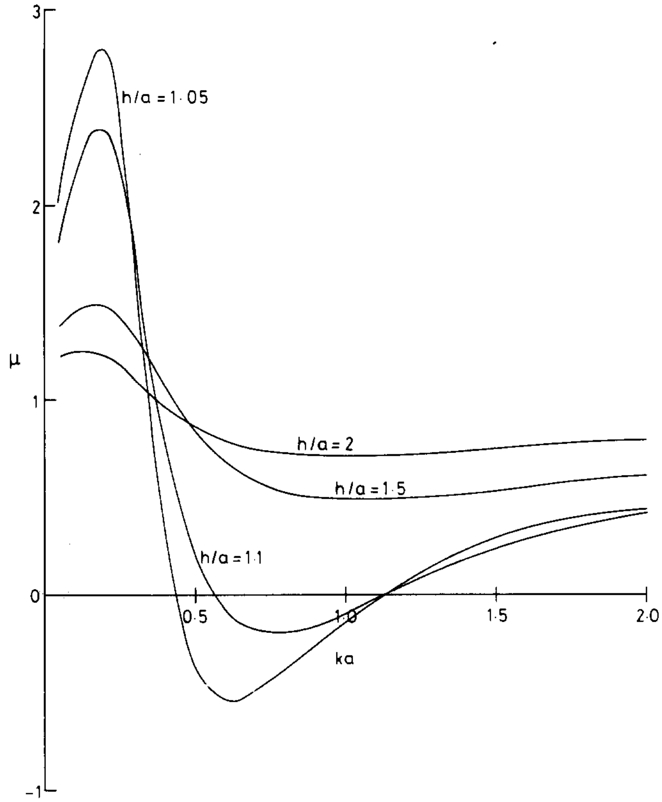

Figure 1. Added mass μ as a function of wavenumber ka for a horizontal cylinder of radius a, depth of submergence h.

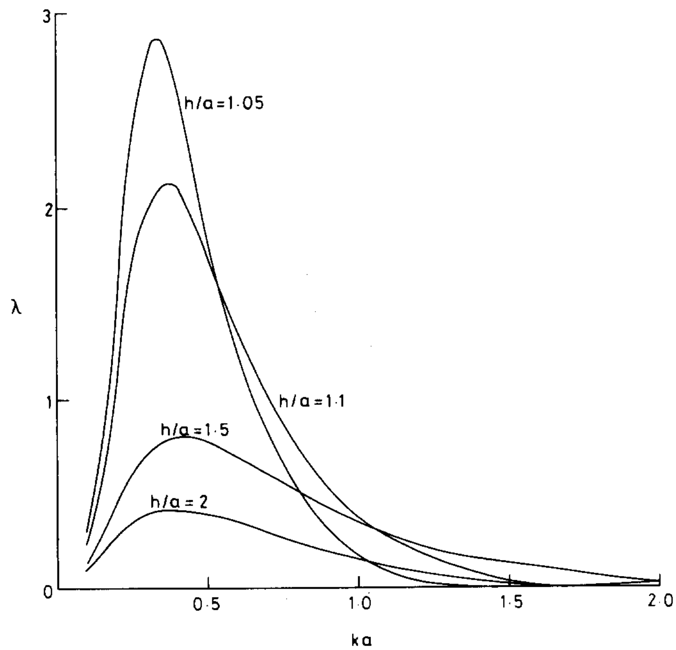

Figure 2. Damping λ as a function of wavenumber ka for a horizontal cylinder of radius a, depth of submergence h.

In multi-body problems the added mass takes the form of a matrix, reflecting the fact that the oscillations of one body can create a force on another. The variation with frequency of the off-diagonal terms is invariably oscillatory taking both positive and negative values. The diagonal terms also tend to be more oscillatory in their frequency variation than for a single body. See, for example, Matsui and Tamaki [6]. It is clear that a simple physical interpretation in terms of accelerated fluid mass is not appropriate in complicated multi-body problems where interaction effects are important.

For a single oscillating body without an interior free surface, it appears that negative added mass occurs only when the body is submerged, and then only over a restricted range of frequencies. Thus Newman, Sortland and Vinje [9] have considered the vertical oscillations of a submerged two-dimensional rectangular cylinder and verified the negative added mass measured by Chung [1]. They presented results obtained using a numerical wave-source distribution method and also using an approximate technique involving the matching of a solution for the flow in the shallow region above the cylinder to a deep-water solution valid away from the cylinder.

The present study seeks to extend the understanding of the phenomenon of negative added mass by considering the heave oscillations of a submerged *vertical* circular cylinder. The solution is by the method of matched eigenfunction expansions used in related problems by Yeung [15] and Thomas [13]. Negative added-mass coefficients are found to occur when the cylinder oscillates close to the free surface and, as in other examples, these are accompanied by rapid variations with frequency in both the added-mass and damping coefficients. An approximate solution valid for long waves in shallow water is used to help interpret the results. In Appendix I a relation due to Falnes (unpublished note) between the kinetic and potential energy of the fluid, and the added mass, is presented, whilst in Appendix II the Kramers-Kronig relations are used to confirm some of the trends apparent in the added-mass and damping curves as functions of frequency.

2. Formulation

A bottom-mounted vertical cylinder of height d and radius a in water of depth h_2 ($d < h_2$) makes time-harmonic oscillations in heave with radian frequency ω and amplitude ζ. The origin of the vertical z-axis is at the mean level of the top face of the cylinder so that the bottom is at $z = -d$ and the mean free-surface level at $z = h_1$ as in Fig. 3. Polar coordinates (r, θ) are chosen in the horizontal plane with the origin at the cylinder axis. Since the motion is axisymmetric there is no dependence on θ.

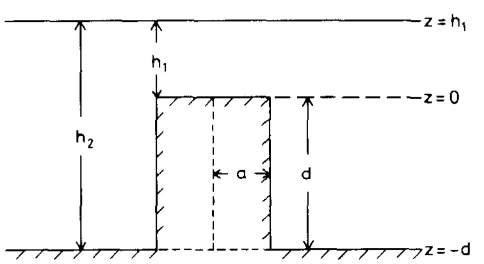

Figure 3. Definition sketch for bottom-mounted vertical cylinder.

Under the usual assumptions of linearised water-wave theory the velocity potential $\Phi(r, z, t)$ for the fluid motion may be written

$$\Phi = \mathrm{Re}\{-i\omega\zeta\phi(r, z)\,\mathrm{e}^{-i\omega t}\}. \tag{2.1}$$

so that the complex-valued spatial potential $\phi(r, z)$ satisfies Laplace's equation within the fluid,

$$\begin{aligned} \nabla^2\phi = 0, \qquad & -d \leqslant z \leqslant h_1, \qquad && r \geqslant a, \\ & 0 \leqslant z \leqslant h_1, && r \leqslant a; \end{aligned} \tag{2.2}$$

the linearised free-surface conditions

$$\frac{\partial\phi}{\partial z} = \frac{\omega^2}{g}\phi, \qquad z = h_1; \tag{2.3}$$

the condition of no flow through the horizontal bottom

$$\frac{\partial\phi}{\partial z} = 0, \qquad z = -d, \qquad r \geqslant a; \tag{2.4}$$

and the conditions on the cylinder surface

$$\frac{\partial\phi}{\partial z} = 1, \qquad z = 0, \qquad r \leqslant a; \tag{2.5}$$

and

$$\frac{\partial\phi}{\partial r} = 0, \qquad -d \leqslant z \leqslant 0, \qquad r = a. \tag{2.6}$$

In addition, only outgoing waves exist as $r \to \infty$.

The solution procedure follows that of Yeung [15] who treated the problem of a floating circular cylinder. The velocity potential is written in terms of appropriate eigenfunctions for the interior $(0 \leqslant z \leqslant h_1,\ r \leqslant a)$ and exterior $(-d \leqslant z \leqslant h_1,\ r \geqslant a)$ regions of the fluid domain and the two expansions matched on $r = a$. The solution of Laplace's equation for the interior region satisfying conditions (2.3) and (2.5) is

$$\phi^{(1)}(r, z) = \sum_{n=0}^{\infty} A_n h_1 \frac{I_0(\alpha_n r)}{I_0(\alpha_n a)} Z_n^{(1)}(z) + z - h_1 + \frac{g}{\omega^2} \tag{2.7}$$

where I_0 is the modified Bessel function of the first kind of order zero, the eigenvalues α_n are solutions of

$$\frac{\omega^2}{g} + \alpha \tan \alpha h_1 = 0, \tag{2.8}$$

and the vertical eigenfunction is

$$Z_n^{(1)}(z) = N_n^{(1)-1/2} \cos \alpha_n z \tag{2.9}$$

with the normalising factor

$$N_n^{(1)} = \frac{1}{2}\left(1 + \frac{\sin 2\alpha_n h_1}{2\alpha_n h_1}\right). \tag{2.10}$$

The first eigenvalue $\alpha_0(= -ik_1)$ is pure imaginary while the remainder are real and positive and taken in order of increasing magnitude. The solution for the exterior region satisfying condition (2.4) is

$$\phi^{(2)}(r, z) = \sum_{n=0}^{\infty} \frac{B_n}{\beta_n} \frac{K_0(\beta_n r)}{K_0'(\beta_n a)} Z_n^{(2)}(z) \tag{2.11}$$

where K_0 is the modified Bessel function of the second kind of order zero and the eigenvalues $\beta_n(\beta_0 = -ik_2)$ satisfy

$$\frac{\omega^2}{g} + \beta \tan \beta h_2 = 0 \tag{2.12}$$

and

$$Z_n^{(2)}(z) = N_n^{(2)-1/2} \cos \beta_n (z + h_1) \tag{2.13}$$

where

$$N_n^{(2)} = \frac{1}{2}\left(1 + \frac{\sin 2\beta_n h_2}{2\beta_n h_2}\right). \tag{2.14}$$

Continuity of fluid pressure and vertical velocity across $r = a$ require that the expressions (2.7) and (2.11) for $\phi^{(1)}$ and $\phi^{(2)}$ be matched on $r = a$, $0 \leqslant z \leqslant h_1$. If the resulting equation is multiplied throughout by $Z_k^{(1)}(z)$ and integrated over $(0, h_1)$, the following result is obtained:

$$A_k = \frac{h_2}{h_1} \sum_{m=0}^{\infty} B_m T_m C_{mk} - A_k^* \tag{2.15}$$

where

$$T_m = \frac{1}{\beta_m h_2} \frac{K_0(\beta_m a)}{K_0'(\beta_m a)}, \tag{2.16}$$

$$C_{mk} = \frac{1}{h_1} \int_0^{h_1} Z_m^{(2)}(z) Z_k^{(1)}(z) \mathrm{d}z, \tag{2.17}$$

and

$$A_k^* = \frac{1}{h_1^2}\int_0^{h_1}\left(z - h_1 + \frac{g}{\omega^2}\right)Z_k^{(1)}(z)\mathrm{d}z. \tag{2.18}$$

Similarly the continuity of the horizontal fluid velocity across $r = a$ requires the matching of $\partial\phi^{(1)}/\partial r$ and $\partial\phi^{(2)}/\partial r$ on $r = a$, $0 \leqslant z \leqslant h_1$. If the resulting equation and the expanded form of Eqn. (2.6) are multiplied by $Z_j^{(2)}(z)$, and then integrated over their respective ranges of validity the result is

$$\sum_{n=0}^{\infty} A_n S_n C_{jn} = \frac{h_2}{h_1} B_j \tag{2.19}$$

where

$$S_n = \alpha_n h_1 \frac{I_0'(\alpha_n a)}{I_0(\alpha_n a)}.$$

Eliminating B_m from Eqn. (2.15) using Eqn. (2.19) gives an infinite system of simultaneous equations for the $A_n's$, viz.

$$A_k - \sum_{n=0}^{\infty} A_n D_{kn} = -A_k^* \tag{2.20}$$

where

$$D_{kn} = S_n \sum_{m=0}^{\infty} T_m C_{mk} C_{mn}. \tag{2.21}$$

Now the vertical hydrodynamic force on the top face (S) of the cylinder is

$$\begin{aligned} F(t) &= \int_S\left(-\rho\frac{\partial\Phi}{\partial t}\right)\mathrm{d}S \\ &= -A\dot{U} - BU, \quad \text{say} \end{aligned} \tag{2.22}$$

where

$$U(t) = \mathrm{Re}(-i\omega\zeta\,\mathrm{e}^{-i\omega t})$$

is the vertical velocity of the cylinder and A and B, dependent on the frequency of oscillation, are the added-mass and damping coefficients describing the components of the force in phase with the acceleration and velocity respectively. Suitable non-dimensional coefficients may be defined by

$$A = M\mu, \qquad B = M\omega\lambda \tag{2.23}$$

where $M = \rho\pi a^3$ so that, after substitution for the interior potential, Eqn. (2.7),

$$\mu + i\lambda = \frac{h_1}{a}\left(1 - \frac{g}{\omega^2 h_1}\right) - 2\sum_{n=0}^{\infty} \frac{A_n S_n}{N_n^{(1)1/2}(\alpha_n a)^2}. \tag{2.24}$$

3. Approximate solution for shallow water

Following Yeung [15] we define depth-averaged velocity potentials

$$\bar{\phi}^{(1)}(r) = \frac{1}{h_1}\int_0^{h_1} \phi^{(1)}(r, z)\mathrm{d}z \tag{3.1}$$

and

$$\bar{\phi}^{(2)}(r) = \frac{1}{h_2}\int_{-d}^{h_1} \phi^{(2)}(r, z)\mathrm{d}z. \tag{3.2}$$

If the boundary conditions (2.3)–(2.6) are used, it follows that these potentials may be shown to satisfy

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\bar{\phi}^{(1)}}{\partial r}\right) + \frac{\omega^2}{gh_1}\bar{\phi}^{(1)} = \frac{1}{h_1} \tag{3.3}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\bar{\phi}^{(2)}}{\partial r}\right) + \frac{\omega^2}{gh_2}\bar{\phi}^{(2)} = 0. \tag{3.4}$$

In deriving Eqns. (3.3) and (3.4), it has been assumed that $\phi^{(i)}(r, h_1)$ may be replaced by $\bar{\phi}^{(i)}(r)$ $(i = 1, 2)$ so that the flow varies little throughout the depth. This is the shallow-water approximation for long waves.

Equation (3.3) has the solution

$$\bar{\phi}^{(1)}(r) = CJ_0(k_1 r) + \frac{g}{\omega^2} \tag{3.5}$$

and Eqn. (3.4) the solution

$$\bar{\phi}^{(2)}(r) = DH_0(k_2 r), \tag{3.6}$$

where

$$\frac{\omega^2}{g} = k_1^2 h_1 = k_2^2 h_2 \tag{3.7}$$

and J_n and H_n are respectively the Bessel function and the Hankel function of the first kind of order n. The solution $\bar{\phi}^{(1)}$ represents an axisymmetric standing wave above the heaving mount while $\bar{\phi}^{(2)}$ is an axisymmetric progressive wave radiating energy away from the mount. The complex constants C and D are determined by demanding that the velocity potentials and the total flow are continuous at $r = a$. This gives

$$C = \frac{\left(k_1^2 h_1\right)^{-1} H_1(\varepsilon k_1 a)}{F(k_1 a)} \tag{3.8}$$

and

$$D = \frac{\varepsilon\left(k_1^2 h_1\right)^{-1} J_1(k_1 a)}{F(k_1 a)}, \tag{3.9}$$

where

$$F(k_1 a) = \varepsilon J_1(k_1 a) H_0(\varepsilon k_1 a) - J_0(k_1 a) H_1(\varepsilon k_1 a) \tag{3.10}$$

and

$$\varepsilon = \frac{k_2}{k_1} = \left(\frac{h_1}{h_2}\right)^{1/2} \tag{3.11}$$

The added-mass and damping coefficients follow from Eqn. (2.22) and, with the non-dimensionalisation of Eqns. (2.23), are given by

$$\mu + i\lambda = -\frac{g}{\omega^2 a}\left(1 + \frac{2 H_1(\varepsilon k_1 a) J_1(k_1 a)}{k_1 a F(k_1 a)}\right). \tag{3.12}$$

4. Results and discussion

To solve the system of Eqns. (2.20) it must be truncated at a finite value N_1 and similarly the summation in Eqn. (2.21) requires truncation at some value N_2. The values N_1 and N_2 correspond to the number of real eigenvalues used in the interior and exterior regions. Detailed examination of a number of trial calculations indicated that a large value of N_2, but only a moderate value of N_1, is required. In all of these trial calculations the added mass and damping were determined to within 2% by taking $N_1 = 10$ and $N_2 = 50$. Convergence was most rapid when either, or both, of h_1/h_2 and a/h_2 were large (i.e. $h_1/h_2 \geqslant 0.5$, $a/h_2 \geqslant 1.0$). All the calculations now presented were made with $N_1 = 20$, $N_2 = 60$.

Figures 4 and 5 give respectively the added-mass and damping coefficients for a cylinder of fixed radius $a/h_2 = 0.1$ at various depths of submergence. These curves resemble those presented by Figs. 1 and 2. For large h_1/h_2 there is little variation in μ and λ with frequency, but with decreasing h_1/h_2 the curves become more peaked and when the mount is sufficiently close to the surface the added mass becomes negative over a range of frequencies. As an extreme example of this Fig. 6 displays added-mass and damping curves for $a/h_2 = 5.0$, $h_1/h_2 = 0.05$.

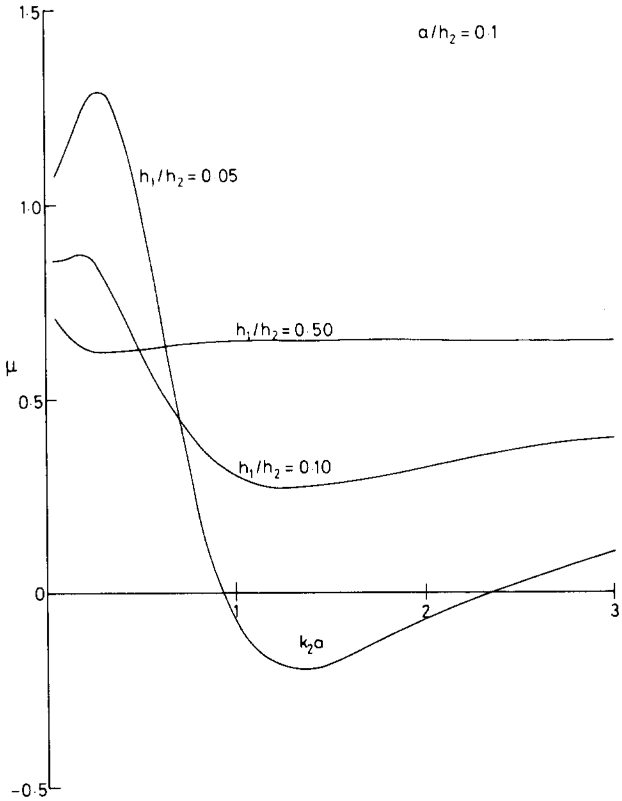

Figure 4. Added mass μ as a function of exterior wavenumber k_2a for a bottom-mounted vertical cylinder of radius $a/h_2 = 0.1$.

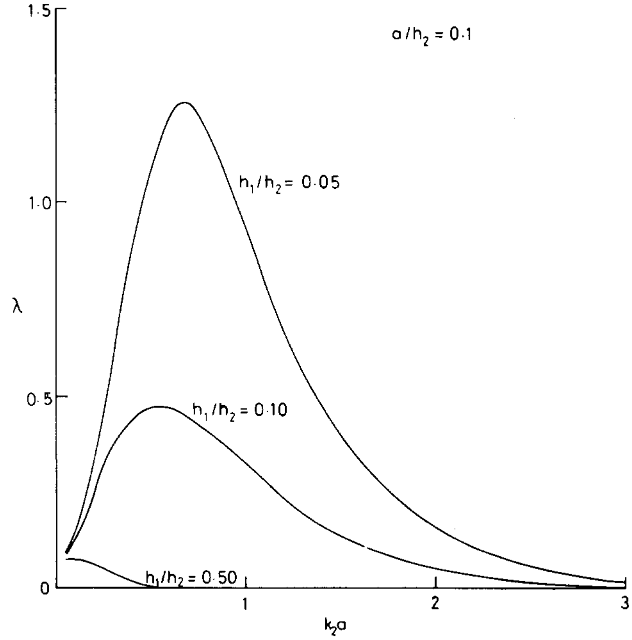

Figure 5. Damping λ as a function of exterior wavenumber k_2a for a bottom-mounted vertical cylinder of radius $a/h_2 = 0.1$.

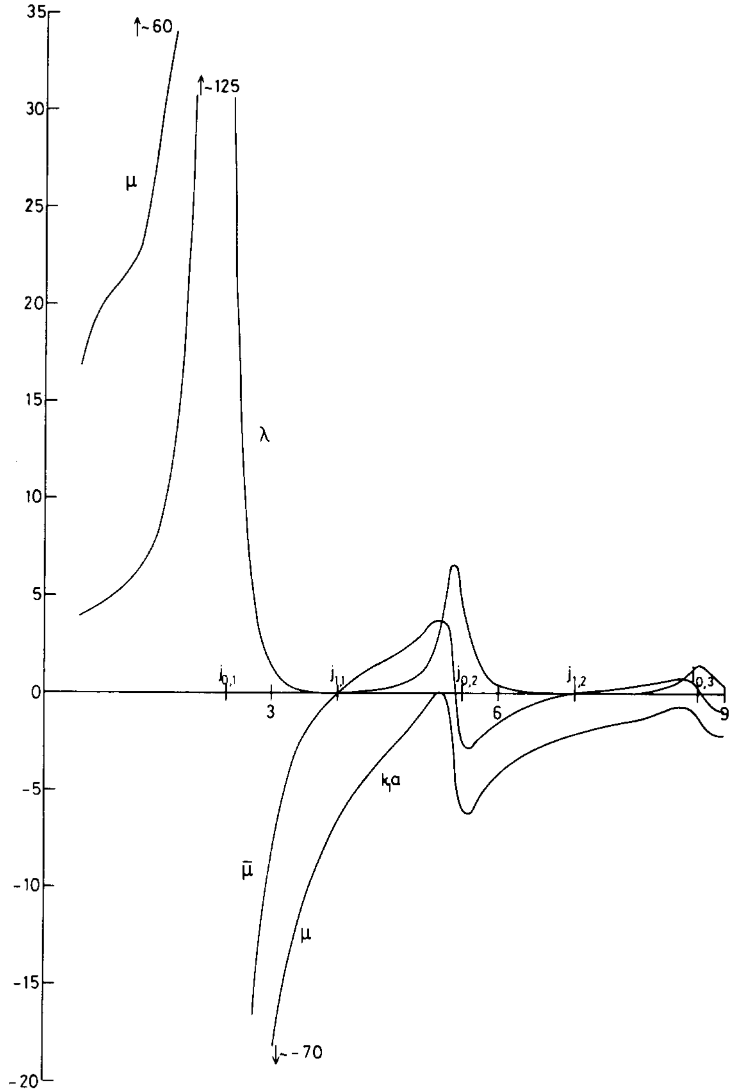

Figure 6. Added mass μ, modified added mass $\bar{\mu}$ and damping λ as function of interior wavenumber k_1a for a bottom-mounted vertical of radius $a/h_2 = 5.0$ and depth of submergence $h_1/h_2 = 0.05$. The numbers at the extremeties indicate the approximate height of each maximum or minimum. The descending portion of the added-mass curve around $k_1a = 2$ has been omitted.

Before considering these results further the contribution of the term

$$z - h_1 + \frac{g}{\omega^2}$$

to the interior potential (Eqn. (2.7)) will be discussed. This represents the rigid-body motion of the fluid above the mount. For this motion alone the added-mass coefficient is given by

$$\mu = \frac{h_1}{a} - \frac{g}{\omega^2 a}, \tag{4.1}$$

(cf. Eqns. (2.24) and (3.12)) while the damping coefficient is identically zero (there is no flow into the outer region to generate waves). It should be noted that for the bottom-mounted cylinder considered here there is a contribution to the force from the fluctuating hydrostatic pressure (relative to the mean surface level). The corresponding force coefficient is equal and opposite to the term $-g/\omega^2 a$ in μ. However, for a body totally surrounded by fluid, as would usually be the case, there would be a cancelling hydrostatic force on the base and so, for the purposes of this discussion, hydrostatic effects will not be considered.

From the numerical results, it has been observed that for high frequencies ($k_1 a \gg 1$) and small depths of submergence equation (4.1) gives the dominant contribution to the added mass. Indeed as $\omega^2 a/g \to \infty$, μ tends to the constant value h_1/a. At low frequencies the contribution to the added mass is dominated by the term $-g/\omega^2 a$, but, in general, the added mass does not become negative because of the wave motion contribution. However, for sufficiently small depths of submergence h_1, the added mass, as given by Eqn. (4.1), will become negative at frequencies where wave effects are negligible. For this reason a second added-mass curve, with the rigid-body-motion contribution taken out and denoted by $\bar{\mu}$, is drawn in Fig. 6. In this way the effects of the resonant wave motion may be isolated.

Falnes (unpublished note) has related the added mass to the energy of the fluid motion. For a body oscillating in a single mode of motion with velocity amplitude U, the added mass A (see Eqn. 2.22)) satisfies

$$T - V = \tfrac{1}{4} A U^2, \tag{4.2}$$

where T and V are respectively the kinetic and potential energies of the total fluid motion averaged over a period. A proof of this result is given in Appendix I. At large depths of submergence (so that the effect of the free surface, and hence V, is negligible) the added mass A may be interpreted as the mass of fluid accelerated by the motion of the body (hence the origin of the term). For a body close to the free surface no simple interpretation of added mass has previously been available. As an illustration of Eqn. (4.2) consider the rigid-body motion of the fluid mass above the heaving cylindrical mount. The mean kinetic energy of the fluid motion is

$$T = \tfrac{1}{4} \rho \pi a^2 h_1 U^2 \tag{4.3}$$

and the mean potential energy is

$$V = \tfrac{1}{4} \rho \pi a^2 \frac{g}{\omega^2} U^2. \tag{4.4}$$

Substitution of these expressions into Eqn. (4.2) and appropriate non-dimensionalisation recovers Eqn. (4.1). As the depth of submergence h_1 decreases so does T because of the reduced mass of fluid above the mount. However, V depends only on the motion of the free surface and is independent of h_1, so that negative added mass occurs when the body is oscillating sufficiently close to the free surface.

From Eqn. (4.2), zeros in the added mass will occur whenever the mean kinetic and potential energies are equal. In Fig. 6 zeros in $\bar{\mu}$ occur at values of $k_1 a$ close to the zeros of $J_1(k_1 a)$ (the nth zero of $J_m(k_1 a)$ is denoted by $j_{m,n}$). Examination of the shallow-water solution, Eqn. (3.5), shows that at these frequencies $\partial\bar{\phi}^{(1)}/\partial r$ is zero and hence there is no flow across the rim of the cylinder into the exterior region. The wave motion (i.e. excluding the rigid body motion) consists entirely of a standing wave confined to the top of the mount; it is a simple calculation to show that the mean kinetic and potential energies of such a motion are equal, hence the zero in $\bar{\mu}$. There is no wave generation in the exterior region and therefore the damping coefficient λ is also zero at these frequencies.

The very sharp peaks in the curves displayed in Fig. 6 suggest that there is a resonant motion at certain frequencies. The shallow-water solution for the added mass and damping (Eqn. (3.12)) shows that resonance will occur at the zeros of $F(k_1a)$. The zeros of this function have been examined in detail by Longuet-Higgins [5] and Summerfield [12]. There are no real zeros for non-zero ε, but for small ε the complex zeros each have a small imaginary part, and rapid variations in the solution, as a function of k_1a, can be expected near the frequencies given by the real part. These frequencies occur near the zeros of $J_0(k_1a)$. Away from the resonant frequencies, but still for small ε, the expression (3.12) reduces to

$$\mu + i\lambda \sim -\frac{g}{\omega^2 a}\left(1 - \frac{2J_1(k_1a)}{k_1aJ_0(k_1a)}\right). \tag{4.5}$$

From Eqn. (4.5) it is apparent that there are zeros in the modified added mass $\bar{\mu}$ at the zeros of $J_1(k_1a)$ with negative values occurring at frequencies in a range below each of the zeros. It can also be seen that the damping is close to zero away from the resonant frequencies. The full shallow-water expression, Eqn. (3.12), gives results very close to the curves in Fig. 6.

At the resonant frequencies of the shallow-water solution ($k_1a = j_{o,n}$), the nodes of the dominant standing-wave component of the complete interior potential (Eqn. (2.7)) are above the rim of the cylinder and there is a non-zero flow velocity into the outer region. The resonant behaviour therefore gives very strong wave generation in the outer region, hence the sharp peaks in the damping curve near the resonant frequencies. From Fig. 6 it can be seen that there is a zero of $\bar{\mu}$ associated with each maximum in λ. The shallow-water solution (3.12) shows that they do not coincide, except as $\varepsilon \to 0$ when they both occur at the resonant wavenumbers $j_{o,n}$. Each maximum in λ is accompanied by a rapid decrease in μ. In Appendix II one of the Kramers-Kronig relations is used to demonstrate that it is generally true that any sharp isolated maximum in λ must be accompanied by such a rapid drop in μ as the frequency increases.

5. Conclusion

The occurrence of negative added mass has been investigated for the heave oscillations of a submerged cylindrical mount. In common with previous work for other submerged bodies, the added mass may become negative when the depth of submergence is small and free-surface effects are important. This is reflected in a new relation (due to Falnes) between the added mass and the mean kinetic and potential energies of the fluid motion (Eqn. (4.2)). At large depths of submergence the mean potential energy of the motion is negligible and the added mass is necessarily positive. At small depths of submergence the oscillations of the free surface are such that the mean potential energy can exceed the mean kinetic energy resulting in negative added mass. For these small submergences near-resonant standing waves may occur above the mount accounting for the rapid changes observed in both added mass and damping.

Acknowledgements

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Appendix I

A relation between the added mass and the energy of the fluid motion due to Falnes (unpublished note)

Consider a body oscillating in a single mode of motion with velocity amplitude U and velocity potential $\Phi = U\phi$. The added mass and damping will then be given by

$$A + iB = \rho \iint_{S_B} \phi \frac{\partial \phi^*}{\partial n} \mathrm{d}S \tag{A1.1}$$

so that, as in Falnes [3],

$$A = \tfrac{1}{2}\rho \iint_{S_B} \frac{\partial}{\partial n}(\phi^*\phi)\mathrm{d}S, \tag{A1.2}$$

$$B = -\frac{i\omega\rho}{2} \iint_{S_B} \left(\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n}\right)\mathrm{d}S. \tag{A1.3}$$

Here S_B is the body surface, n is the normal to the surface directed out of the fluid and * indicates complex conjugate.

Application of Green's theorem to ϕ and ϕ^* and the use of (A1.3) gives

$$B = i\omega\rho \iint_{S_\infty} \phi \frac{\partial \phi^*}{\partial n} \mathrm{d}S, \tag{A1.4}$$

where the integration is carried out over a surface S comprising S_B, the free surface S_F, the horizontal bottom and an enclosing cylinder S_∞ in the far field.

The mean potential energy of the fluid motion over a period is

$$\begin{aligned} V &= \iint_{S_F} (\tfrac{1}{4}\rho g|\eta|^2)\mathrm{d}S \\ &= \tfrac{1}{4}\rho \frac{\omega^2}{g} U^2 \iint_{S_F} (\phi\phi^*)\mathrm{d}S, \end{aligned} \tag{A1.5}$$

where η is the displacement of the free surface from the mean level. The mean kinetic energy is

$$T = \tfrac{1}{4}\rho U^2 \iiint_\tau (\nabla\phi \cdot \nabla\phi^*)\mathrm{d}\tau \tag{A1.6}$$

where τ denotes the fluid volume. By the divergence theorem

$$T = \tfrac{1}{4}\rho U^2 \iint_S \phi \underline{n} \cdot \nabla\phi^* \mathrm{d}S$$

$$= \tfrac{1}{4}\rho U^2 \left(\iint_{S_B} \phi \frac{\partial\phi^*}{\partial n} \mathrm{d}S + \frac{\omega^2}{g} \iint_{S_F} \phi\phi^* \mathrm{d}S + \iint_{S_\infty} \phi \frac{\partial\phi^*}{\partial n} \mathrm{d}S \right). \tag{A1.7}$$

Combining Eqns. (A1.1), (A1.4), (A1.5) and (A1.7) gives

$$T - V = \tfrac{1}{4} A U^2. \tag{A1.8}$$

The generalisation of this result for N modes of motion is

$$T - V = \tfrac{1}{4} \sum_{i,j} A_{ij} U_i U_j^* \tag{A1.9}$$

where U_i is the complex velocity amplitude of the ith mode.

Appendix II

The Kramers-Kronig relation for the added mass in terms of the damping coefficients is

$$\mu(\nu) - \mu(\infty) = \frac{1}{\pi} ⨍_0^\infty \frac{\lambda(z)\mathrm{d}z}{z - \nu}, \tag{A2.1}$$

where $\nu = ka$ (see Kotik and Mangulis [4]). Suppose the damping λ has a sharp maximum at $\nu = \nu_0$. Close to this maximum, at $\nu = \nu_0 + \varepsilon$ where $|\varepsilon|$ is small,

$$\mu(\nu_0 + \varepsilon) - \mu(\infty) = \frac{1}{\pi} ⨍_0^\infty \frac{\lambda(z)}{z - (\nu_0 + \varepsilon)} \mathrm{d}z$$

$$= \frac{1}{\pi} ⨍_{-\nu_0}^\infty \frac{\lambda(\nu_0 + Z)}{Z - \varepsilon} \mathrm{d}Z. \tag{A2.2}$$

The major contribution to the integral will be from around $Z = 0$. We therefore replace the range of integration by a suitable finite range $(-\delta_1, \delta_2)$, $\delta_1, \delta_2 > 0$ such that $|\varepsilon| \ll \delta_1, \delta_2$. Now

$$\lambda(\nu_0 + Z) \sim \lambda(\nu_0) + Z\lambda'(\nu_0) + \tfrac{1}{2}Z^2\lambda''(\nu_0) + \ldots$$
$$= \lambda(\nu_0) + O(Z^2) \tag{A2.3}$$

because $\lambda(\nu)$ has a maximum at $\nu = \nu_0$. Equation (A2.2) can therefore be approximated by

$$\mu(\nu_0 + \varepsilon) - \mu(\infty) = \frac{\lambda(\nu_0)}{\pi} \oint_{-\delta_1}^{\delta_2} \frac{dZ}{Z - \varepsilon}. \tag{A2.4}$$

Carrying out the integration in Eqn. (A2.4) and expanding in powers of ε gives

$$\mu(\nu_0) + \varepsilon\mu'(\nu_0) - \mu(\infty) \sim \frac{\lambda(\nu_0)}{\pi}\left(\ln\frac{\delta_2}{\delta_1} - \frac{\varepsilon(\delta_1 + \delta_2)}{\delta_1\delta_2}\right) \tag{A2.5}$$

and the slope of the added mass curve at $\nu = \nu_0$ is therefore

$$\mu'(\nu_0) = -\frac{\lambda(\nu_0)}{\pi}\frac{\delta_1 + \delta_2}{\delta_1\delta_2} < 0. \tag{A2.6}$$

The sharper the peak in the damping curve then the smaller the values of δ_1, δ_2 required. Hence, a large sharp maximum in the damping will be accompanied by a very rapid drop in the added mass, as a function of frequency.

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